1 a) $f(x) = \sin\frac{x}{4} - \cos\frac{x}{4}$

① $f'(x) = \frac{1}{4}\cos\frac{x}{4} + \frac{1}{4}\sin\frac{x}{4}$

$$x_{n+1} = x_n - \frac{\left(\sin\left(\frac{x_n}{4}\right) - \cos\left(\frac{x_n}{4}\right)\right)}{\frac{1}{4}\cos\left(\frac{x_n}{4}\right) + \frac{1}{4}\sin\left(\frac{x_n}{4}\right)}$$

$x_0 = 3 \qquad x_1 = 3.14165182 \qquad$ exact err $5.917\ 10^{-5}$

ERROR estimate : $\frac{x_1 - x_0}{x_1} = 0.045088$

② $f(x)$ increasing

$m_0 = 3 \qquad f(m_0) = -0.05 < 0 \qquad [a_1, b_1] = [3, 4]$

$m_1 = 3\frac{1}{2} \qquad f(m_1) = 0.1265 > 0 \qquad [a_2, b_2] = [3, 3\frac{1}{2}]$

$m_2 = 3\frac{1}{4} \qquad$ err $< \frac{1}{4} \qquad \left(\frac{1}{2}\right)^n \cdot \frac{1}{4} < 0.045088 \qquad n \geq 3$ extra iterations

③ 2nd iteration Newton gives fast increase of accuracy (2nd order)
Bisection only gives linear reduction of errors

b) $g(x) = x + \alpha\cos\left(\frac{x}{4}\right) - \alpha\sin\left(\frac{x}{4}\right)$

$g'(x) = 1 - \frac{\alpha}{4}\sin\left(\frac{x}{4}\right) - \frac{\alpha}{4}\cos\left(\frac{x}{4}\right) \qquad \alpha = 1$ at first

① $K = g'(\pi) = 1 - \frac{\alpha}{4}\frac{\sqrt{2}}{2} - \frac{\alpha}{4}\frac{\sqrt{2}}{2} = 1 - \alpha\frac{\sqrt{2}}{4} \longrightarrow 1 - \frac{\sqrt{2}}{4} = 0.64644661$

$\hat{K} = \frac{x_4 - x_3}{x_3 - x_2} = 0.64647271$

err reduction $\frac{x_4 - \pi}{x_3 - \pi} = 0.64645215 \longrightarrow$ almost equal

② err est $= \frac{K}{1-K}|x_4 - x_3| = 0.02473393$ almost equal

true err $= |x_4 - \pi| = 0.02473170$

③ $x_4 - \frac{(x_4 - x_3)^2}{x_4 - 2x_3 + x_2} = 3.14159408$

(notice : error $= 2.225\ 10^{-6}$)

④ $g'(\pi) = 1 - \alpha\frac{\sqrt{2}}{4} \qquad -1 < 1 - \alpha\frac{\sqrt{2}}{4} < 1 \Rightarrow \left[0, \frac{8}{\sqrt{2}}\right]$

optimal for $1 - \alpha\frac{\sqrt{2}}{4} = 0 \Rightarrow \alpha = \frac{4}{\sqrt{2}}$

2 a) $x_0 \quad 2h \quad x_2$ $\qquad T(2h) = \frac{2h}{2}\left(f(x_0) + f(x_2)\right)$

$h \quad h$

① $x_0 \quad x_1 \quad x_2$ $\qquad T(h) = \frac{h}{2}\left(f(x_0) + f(x_1)\right) + \frac{h}{2}\left(f(x_1) + f(x_2)\right)$

$$\frac{4}{3}T(h) - \frac{1}{3}T(2h) = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$ equal

$$S(2h) = \frac{2h}{6}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

② $h=1$; $0 \quad \frac{1}{2} \quad 1$
$$S = \frac{1}{6}\left(e^{\sqrt{0}} + 4e^{\sqrt{1/2}} + e^{\sqrt{1}}\right)$$
$$= \frac{1}{6}\left(1 + 4e^{\sqrt{1/2}} + e\right)$$

③ $f(x) = e^{\sqrt{x}} \rightarrow f'(x) = e^{\sqrt{x}}\frac{1}{2\sqrt{x}} \rightarrow f^{(4)}$ has $\frac{1}{x^a}$ behaviour $a>0$

unbounded for $x=0$ $\longrightarrow$ err est $\underset{\text{says}}{:}$ $\varepsilon < \infty$ useless

b) ① $q = \frac{I_{16} - I_{32}}{I_{32} - I_{64}}$ for I_1: $q = 2.7419$
I_2: $q = 3.9995$

Trapezium for I_1 not optimal
Because of sqrt. Trapezium for I_2 optimal (2^{nd} order)

② $\left|\frac{I_{64} - I_{32}}{3}\right| = 1.8052 \; 10^{-4}$

③

$I(4)$		
$I(8)$	$\frac{4}{3}I_8 - \frac{1}{3}I_4 = \overset{T_2(8)}{2.00002129}$	
$I(16)$	$\frac{4}{3}I_{16} - \frac{1}{3}I_8 = \underset{T_2(16)}{2.00000133}$	$\frac{16}{15}T_2(16) - \frac{1}{15}T_2(8) = 2.0000\,0000$

error $= 8.9 \; 10^{-10}$

④ $\left(\frac{1}{4}\right)^n 1.8052 \; 10^{-4} < 8.9 \; 10^{-10}$ $\qquad n \geq 9$ times halving gridsize

$\Rightarrow 64 * 2^9 = 2^{15}$ segments
$(= 32768)$

3 a) $y_{n+1} = y_n + h\left(\frac{1}{y_n} - x_n\right)$

① $y(\tfrac{1}{2}) = 1 + \tfrac{1}{2}\left(\tfrac{1}{1} - 0\right) = 1\tfrac{1}{2}$

$y(1) = 1\tfrac{1}{2} + \tfrac{1}{2}\left(\frac{1}{1\tfrac{1}{2}} - \tfrac{1}{2}\right) = 1\tfrac{7}{12}$ $(= 1.5833\ldots)$

② $k_1 = \tfrac{1}{2}\left(\tfrac{1}{1} - 0\right) = \tfrac{1}{2}$

$k_2 = \tfrac{1}{2}\left(\frac{1}{1+\tfrac{1}{2}} - \tfrac{1}{2}\right) = \tfrac{1}{12}$

$y(\tfrac{1}{2}) = 1 + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{12}\right) = 1\tfrac{7}{24}$ $(= 1.2916\ldots)$

③ $y_{n+1} = y_n + h\left(\frac{1}{y_{n+1}} - x_{n+1}\right)$

$y_{n+1} = 1 + 1\left(\frac{1}{y_{n+1}} - 1\right) \Rightarrow y_{n+1} = \frac{1}{y_{n+1}} \Rightarrow y_{n+1} = 1$ (or $y_{n+1} = -1$)

b) $q = \left|\frac{y_{960} - y_{1920}}{y_{1920} - y_{3840}}\right| = 8.3332\ldots$

① close to 2^3, according to theory, 3^{rd} order

② $y(x) = y_h(x) + \alpha h^3 + O(h^4)$

$y(x) = y_{\frac{h}{2}}(x) + \alpha \frac{h^3}{8} + O(h^4)$

$\Rightarrow y(x) = \frac{8}{7} y_{\frac{h}{2}}(x) - \frac{1}{7} y_h(x) + O(h^4)$

extrap $= \frac{8}{7} y_{3840}^{(x=5)} - \frac{1}{7} y_{1920}^{(x=5)} = 0.\underline{2016690116}474(57)$

③ err est $= \frac{1}{7}\left(y_{3840}^{(x=6)} - y_{1920}^{(x=6)}\right) = \frac{1}{7}(4.154\ 10^{-10}) = 5.9343\ 10^{-11}$

largest at $x=6$

$8^n \cdot 5.9343\ 10^{-11} < 5.0\ 10^{-4} \Rightarrow n \leq 7$ grid can be $\frac{3840}{2^7} = 30$ segments

but for stability $N=60$ (see ④) → no problems when $N=60$ (stable, sufficient accuracy)

④ wiggles for $N=30$, solution smooth for $N=60$, solution similar to $N \geq 960$

stability limit between $N=30$ and $N=60$

4 a)

$$\underset{x_1}{\overset{0}{|}}\;—\;\underset{x_2}{|}\;—\;\underset{x_3}{|}\;—\;\underset{x_4}{|}\;—\;\underset{x_5}{|}\;—\;\underset{x_6}{\overset{10}{|}} \qquad \Delta x = 2$$

left: $\frac{y_2 - y_1}{\Delta x} = 3$ right: $y_6 = 0$

interior: $\frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} + \frac{2}{10} y_i = \cos\left(\frac{\pi}{2} x_i\right) \quad i = 2..5$

$$\Rightarrow \frac{1}{4} y_{i-1} + \left(-\frac{2}{4} + \frac{2}{10}\right) y_i + \frac{1}{4} y_{i+1} = \cos\left(\frac{\pi}{2} x_i\right)$$

$$\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{4} & -\frac{3}{10} & \frac{1}{4} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{4} & -\frac{3}{10} & \frac{1}{4} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{4} & -\frac{3}{10} & \frac{1}{4} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{4} & -\frac{3}{10} & \frac{1}{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{vmatrix} = \begin{vmatrix} 3 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{vmatrix}$$

b) interior: $\frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} + \frac{y_i - y_{i-1}}{\Delta x} + \frac{2}{10} y_i = \cos\left(\frac{\pi}{2} x_i\right)$

(1)

$$\Rightarrow \underbrace{\left(\frac{1}{(\Delta x)^2} - \frac{1}{\Delta x}\right)}_{\frac{1}{4} - \frac{1}{2}} y_{i-1} + \underbrace{\left(\frac{-2}{(\Delta x)^2} + \frac{1}{\Delta x} + \frac{2}{10}\right)}_{-\frac{2}{4} + \frac{1}{2} + \frac{1}{5}} y_i + \underbrace{\frac{1}{(\Delta x)^2}}_{1/4} y_{i+1} = \cos\left(\frac{\pi}{2} x_i\right)$$

L D (R) values for interior change $\Rightarrow -\frac{1}{4}, \frac{2}{10} \left(\frac{1}{4}\right)$

(2) TDMA has fixed number of operations → takes same time

(3)

$$\underset{x_1}{\overset{0}{|}}\;——\;\underset{x_2}{\overset{5}{|}}\;——\;\underset{x_3}{\overset{10}{|}} \qquad \Delta x = 5$$

$$\begin{vmatrix} -\frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{4}{25} & \frac{8}{25} & \frac{1}{25} \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 3 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \left.\begin{matrix} -1 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} 15 \\ 0 \\ 0 \end{matrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -30 \\ -15 \\ 0 \end{pmatrix}$$

(4) $y'(x_i) = \frac{y_i - y_{i-1}}{\Delta x} + O(\Delta x)$ ← less accurate if $\alpha \neq 0$

$y''(x_i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} + O(\Delta x^2)$

Program Questions : elements of programs/codes

1c) initialisation two values
loop construction
iteration formula Secant
Shift values for efficiency (only 1 function call per iteration)
error estimate $\frac{|x_{n+1} - x_n|}{|x_{n+1}|} < 1E-6$ stop criterion

2c) grid between [0,1]
loop construction
summation midpoint areas
midpoint formula

refinement procedure
err est $\left|\frac{I_{N/2} - I_N}{3}\right| < 1E-6$ stop criterion

efficiency: minimize refinement steps (e.g. divide err est by 4^n)
OR
re-use function values

3c) grid between [0,6]
loop construction over grid
RK2 formulas to go from $y_n \rightarrow y_{n+1}$ (Heun)
err est (whole grid) $\left|\frac{y_{N/2} - y_N}{3}\right| < 1E-6$ stop criterion

refinement procedure (whole grid)

(efficiency not in question)